A CRITERION FOR CELLULARITY IN A MANIFOLD. II

BY

D. R. McMILLAN, JR.[1]

1. **Introduction.** Let X be a compact absolute retract in the interior of a piecewise-linear (abbreviated: pwl) n-manifold M^n. We say that X is *cellular with respect to pwl cells in M^n* if there exists a sequence $F_1, F_2, \ldots$ of pwl n-cells in M^n such that $F_{i+1} \subset \operatorname{Int} F_i$ and $X = \bigcap_{i=1}^{\infty} F_i$. In [12] we showed that, if $n \geq 5$, then X is cellular with respect to pwl cells in M^n if and only if: for each open set U in M^n such that $X \subset U$, there is an open set V in M^n such that $X \subset V \subset U$ and each loop in $V - X$ is contractible in $U - X$. In case $n = 3$ and this condition holds, the proofs of Lemma 1 and Theorem 1' of [12] yield only the unsatisfactory result that X is the intersection of a sequence $F_1, F_2, \ldots$ of homotopy 3-cells, where $F_{i+1} \subset \operatorname{Int} F_i$.

A corollary of the theorems presented here is that if, in the case $n = 3$, X is homeomorphic to a collapsible complex (see [15]), then X is cellular if and only if the above cellularity criterion holds. In particular, our results apply to closed topological cells of dimension three or less. The problem is to show that some neighborhood of X in M^3 is embeddable in E^3. If M^3 is orientable, this reduces to the question of whether polygonal arcs can be deformed away from X with a homeomorphism of M^3.

By a *cube-with-handles* we mean any complex piecewise-linearly equivalent to the regular neighborhood in E^3 of a finite graph. If X is a compact subset of $\operatorname{Int} M^3$, then X has the *cube-with-handles property* if $X = \bigcap_{i=1}^{\infty} H_i$, where H_i is a polyhedral cube-with-handles in M^3 and $H_{i+1} \subset \operatorname{Int} H_i$. It follows from our Corollary 1.3 that collapsible complexes have the cube-with-handles property under every topological embedding in the interior of a 3-manifold. The same is true of 2-manifolds with nonempty boundary, if the 3-manifold M^3 is orientable (see the Remark following Corollary 1.3). We regard Theorem 1 and Corollary 1.2 as our main results, although Corollary 1.3 is perhaps most useful for verifying that subsets have the cube-with-handles property.

In general, we shall use the terminology of [12]. In particular, we retain the convention that a manifold is always connected. All of the manifolds considered will be assumed to have a fixed pwl (i.e., combinatorial) structure. A *closed* manifold is compact and without boundary. An *open* manifold is noncompact and without boundary. By an *n-frame* ($n \geq 2$) we mean any space homeomorphic to the join of

a point with n other distinct points. A *null* sequence $X_1, X_2, \ldots$, of subsets of a metric space is one such that the diameters of the X_i's converge to zero.

Our results lean heavily on the papers of R. H. Bing listed in the references. The reader is referred to [4, §3] for a discussion of Sierpiński curves. A Sierpiński curve in the interior of a 3-manifold M^3 is said to be *tame* if it lies on a tame 2-sphere in M^3. With this understanding, the proofs of Theorems 8.2 and 9.1 of [6] can be adapted to cover the case of a 2-sphere embedded in the interior of a 3-manifold (see [7]).

Finally, we wish to thank R. H. Bing for many helpful discussions concerning the results given here.

2. Tunneling into three-manifolds.

The following lemma tells us how to bore holes in an orientable 3-manifold-with-boundary so as to obtain a cube-with-handles.

LEMMA 1. *Let N be a compact, orientable piecewise-linear 3-manifold with non-empty boundary. Then N contains a finite, disjoint collection of polygonal arcs $A_1, A_2, \ldots, A_k$, such that $A_i \cap \text{Bd } N = \text{Bd } A_i$, for each i, and the closure of N minus a sufficiently small regular neighborhood of each A_i is a cube-with-handles.*

Proof. Since Bd N is nonempty, there is a 2-complex $K \subset N$ such that N contracts (in the sense of J. H. C. Whitehead [15]) to K. We may obtain a 3-manifold equivalent to a regular neighborhood of K in N by attaching, in some manner, disjoint polyhedral 3-cells $B_1, B_2, \ldots, B_k$ (essentially, obtained by thickening the 2-simplexes of K) to a cube-with-handles H (a small regular neighborhood of the 1-skeleton of K) in such a way that each $B_i \cap H = (\text{Bd } B_i) \cap (\text{Bd } H)$ is an annulus. Since the assertion of the lemma clearly holds for a 3-manifold of this form, and since N is combinatorially equivalent to any regular neighborhood of K in N, the result follows.

The following is an application of Lemma 1.

LEMMA 2. *Let X be a compact, connected subset of Int M, where M is an orientable piecewise-linear 3-manifold. Then, X has the cube-with-handles property if and only if the following condition holds:*

For each polygonal arc $A \subset \text{Int } M$ such that $X \cap \text{Bd } A = \varnothing$, and for each $\varepsilon > 0$, there is a homeomorphism h of M onto M such that h is the identity outside the ε-neighborhood of X, and $X \cap h(A) = \varnothing$.

Proof. Suppose first that the condition holds, and let U be an open subset of Int M containing X. Let N be a compact, orientable piecewise-linear 3-manifold with nonempty boundary such that $X \subset \text{Int } N \subset \overline{N} \subset U$. Let $A_1, A_2, \ldots, A_k$, be the arcs promised by Lemma 1. Then, by our condition, there is a homeomorphism $h_1: N \to N$ such that h_1 is the identity on a neighborhood of Bd N and $X \cap h_1(A_1) = \varnothing$. By [1, Theorem 3], we may suppose that h_1 is pwl. Applying the condition inductively gives pwl homeomorphisms $h_1, h_2, \ldots, h_k$ of N onto N such that, for

$2 \leq i \leq k$, h_i is the identity on Bd N and on $h_{i-1}h_{i-2}\cdots h_1(A_1 \cup \cdots \cup A_{i-1})$, and $X \cap h_i[h_{i-1}\cdots h_1(A_i)] = \varnothing$. Then, $h = h_k h_{k-1} \cdots h_1$ is a pwl homeomorphism of N onto N such that h is the identity on Bd N, and $X \cap h(A_i) = \varnothing$, for all i. The closure of N minus a small regular neighborhood of each $h(A_i)$ is a polyhedral cube-with-handles H such that $X \subset \text{Int } H \subset H \subset U$, and the proof is complete.

Now suppose that X has the cube-with-handles property, and let A and $\varepsilon > 0$ be given. Let H be a polyhedral cube-with-handles in the ε-neighborhood of X such that $X \subset \text{Int } H$ and $H \cap \text{Bd } A = \varnothing$. Choose a one-dimensional polyhedron G in $(\text{Int } H) - A$ such that H contracts to G, and let T be a "thin" regular neighborhood of G in $(\text{Int } H) - A$. The desired homeomorphism h is then one which expands T until it exactly covers H, and is the identity outside a small neighborhood of H (the closure of $H - T$ is topologically $(\text{Bd } H) \times [0, 1]$).

The following is probably well known, but we include its proof for completeness.

LEMMA 3. *Let M be a piecewise-linear 3-manifold, A a tame arc in M with $(\text{Int } A) \cap (\text{Bd } M) = \varnothing$, and X a compact subset of Int M. Suppose also that X is either zero-dimensional or a one-dimensional absolute neighborhood retract. Then for given $\varepsilon > 0$, there is a homeomorphism $h: M \to M$ such that $A \cap h(X) = \varnothing$, h moves each point less than ε, and h is the identity outside the ε-neighborhood of $X \cap A$.*

Proof. There is no loss of generality in assuming that $X \cap A$ is a 0-dimensional subset of Int A. Since A is tame, the ε-neighborhood of $X \cap A$ contains a disjoint collection $C_1, C_2, \ldots, C_m$, of tame 3-cells whose interiors cover $X \cap A$, such that each C_j has diameter less than ε and $C_j \cap A$ is an unknotted spanning arc of C_j, for each j. In case X is a 1-dimensional absolute neighborhood retract, we also suppose that the diameter of each C_j is less than δ, where δ is a positive number such that each subset of X of diameter less than δ is contractible to a point in X.

Suppose $A \cap \text{Bd } C_j = a_j \cup b_j$. It then follows from the way that the C_j's were selected that $X \cap \text{Bd } C_j$ cannot separate Bd C_j (since $X \cap \text{Bd } C_j$ is either a compact 0-dimensional set or a compact 1-dimensional set each of whose components is acyclic in the sense of Čech homology). Let B_j be an arc in $(\text{Bd } C_j) - X$ from a_j to b_j. The desired homeomorphism h is then defined to be the identity outside a small neighborhood of each C_j, and to take each B_j onto $C_j \cap A$, leaving $A - \bigcup \text{Int } C_j$ fixed.

COROLLARY. *Let M be an orientable piecewise-linear 3-manifold and X a compact subset of Int M. If X is either zero-dimensional or a one-dimensional absolute neighborhood retract, then for given $\varepsilon > 0$, the ε-neighborhood of X contains a finite disjoint collection of polyhedral cubes-with-handles whose interiors cover X.*

Proof. This follows from Lemma 1, Lemma 3, and the first half of the proof of Lemma 2. In case X is connected, the collection of cubes-with-handles need not contain more than one element, of course.

3. **Pushing arcs away from cells.**   For the reader's convenience, we quote two results from the literature before beginning the proof of Theorem 1. The next lemma follows from [11].

LEMMA 4. *Let D be a 2-cell topologically embedded in the interior of a 3-manifold M. Let $p \in$ Bd D, $q \in$ Int D, and $\varepsilon > 0$. Then there is a tame arc A in D joining a point $r \in$ Bd D to a point $s \in$ Int D such that (Int A) $\cap$ (Bd D) $= \varnothing$, r is within ε of p, and s is within ε of q.*

We shall also need the following useful result of Bing [8, Theorem 2]:

LEMMA 5. *Suppose S is a topological 2-sphere and A is a tame arc in E^3. Then for each $\varepsilon > 0$ there is a homeomorphism $h: E^3 \to E^3$ such that A intersects $h(S)$ at only a finite number of points and pierces it at each of these points, h moves each point less than ε, and h is the identity outside the ε-neighborhood of $S \cap A$. Further, if $p \in A \cap h(S)$, there is an arc Z in S such that $p \in$ Int Z and $A \cup Z$ is a tame 4-frame.*

REMARK 1.   This lemma extends immediately to the case of a finite number of mutually exclusive, tame arcs $A_1, A_2, \ldots, A_k$. We shall also need to use it when the 2-sphere S is replaced by a 2-cell C. This is justified as follows. Use Lemma 3 to shift C slightly until Bd C misses each A_i. Another slight adjustment will make each $A_i \cap C$ a zero-dimensional subset of Int C. Finally, we use the above lemma and the fact that sufficiently small subdisks of Int C can be extended to 2-spheres [2, Theorem 5], to get the desired result.

REMARK 2.   There is a simple closed curve J in Int C, and containing $h(C) \cap \bigcup A_i$, such that $J \cup \bigcup A_i$ is tame. Such a curve may be pieced together from the Z's promised by the last sentence of Lemma 5 and the arcs given by Lemma 4. We show local tameness (and hence tameness, by [1, Theorem 9]) of $J \cup \bigcup A_i$ by appealing to the fact that the union of two tame arcs meeting only at an endpoint is tame if this union lies interior to a 2-cell (this follows from [10, Theorem 3] and [9, Theorem 1]).

REMARK 3.   Similarly, if $h(C) \cap \bigcup A_i$ consists of the points $p_1, p_2, \ldots, p_m$, and E is an arc in Bd $h(C)$, there are mutually exclusive arcs $J_1, J_2, \ldots, J_m$, in $h(C) - E$ such that Bd J_j consists of p_j plus the single point $J_j \cap$ Bd $h(C)$, for each j, and $\bigcup J_j \cup \bigcup A_i$ is tame.

THEOREM 1. *Let B be a q-cell ($q = 1, 2,$ or 3) topologically embedded in E^3, and let $D \subset$ Bd B be a $(q-1)$-cell. Let $A_1, A_2, \ldots, A_k$ be a finite disjoint collection of tame arcs in $E^3 - D$ with each Bd $A_i \subset E^3 - B$. Then, there exists a compact set $C \subset B - D$ such that, for given $\varepsilon > 0$, there is a homeomorphism $h: E^3 \to E^3$ with each $h(A_i) \subset E^3 - B$ and h is the identity outside the ε-neighborhood of C.*

Proof. Let C be a topological q-cell in $B - D$ containing $B \cap \bigcup A_i$ and such that the closure of $B - C$ is a q-cell meeting C in a $(q-1)$-cell E missing $\bigcup A_i$. We assume

that there is given a positive number ε less than the distance from C to D and from $\bigcup A_i$ to the closure of $B-C$.

1. *The case* $q=1$. This follows immediately from Lemma 3.

2. *The case* $q=2$. By the first remark following Lemma 5, we may assume without loss of generality that each A_i has been adjusted slightly so that $B \cap \bigcup A_i$ is a finite set of points $p_1, p_2, \ldots, p_m$, in Int C such that, for each j, there is some arc in $\bigcup A_i$ which pierces B at p_j. By the third remark following Lemma 5, there are mutually exclusive arcs $J_1, J_2, \ldots, J_m$, in $C-E$ such that Bd J_j consists of p_j plus the single point $J_j \cap$ Bd C, for each j, and $\bigcup J_j \cup \bigcup A_i$ is tame.

In particular, since J_j plus some arc Y_j of diameter less than ε in $\bigcup A_i$ is a tame 3-frame, there are mutually exclusive tame 3-cells $F_1, F_2, \ldots, F_m$, in the ε-neighborhood of C and missing the closure of $B-C$ such that $Y_j = F_j \cap \bigcup A_i, J_j \subset$ Int F_j, and Y_j is an unknotted spanning arc of F_j. Suppose Bd $Y_j = a_j \cup b_j$.

We claim that for each j there is an arc R_j in Bd F_j joining a_j and b_j, and missing C. If not, then some component K of $C \cap$ Bd F_j separates a_j from b_j in Bd F_j.

If L is a simple closed curve formed by joining a_j and b_j with an arc in E^3-B whose interior misses F_j, then, since $K \subset C-\bigcup J_j$ and $C-\bigcup J_j$ is simply-connected, there is a neighborhood U of K in E^3 such that each simple closed curve in U is contractible in E^3-L. But $U \cap$ Bd F_j is a neighborhood of K in Bd F_j and so must contain a simple closed curve separating a_j from b_j (since K is the intersection of a monotonically decreasing sequence of punctured 2-cells in Bd F_j), and this simple closed curve must link L (in the homology sense), a contradiction. Hence, the arc R_j exists, for each j.

The desired homeomorphism h is now defined as one which is the identity outside a small neighborhood of each F_j, and which, for each j, takes Y_j onto R_j leaving a_j and b_j fixed.

3. *The case* $q=3$. Let $S=$ Bd C. By Lemma 5, we may assume that each A_i has been adjusted slightly so that $S \cap \bigcup A_i$ is a finite set of points $p_1, p_2, \ldots, p_m$, in $S-E$ and that, for each j, some arc in $\bigcup A_i$ pierces S at p_j. By the second remark following Lemma 5, if we let F denote the closure of $S-E$, there is a simple closed curve $J \subset$ Int F which contains $p_1, p_2, \ldots, p_m$, and such that $J \cup \bigcup A_i$ is tame.

By [6, Theorem 9.1], there is a tame Sierpiński curve X in S such that each point of J is an inaccessible point of X. Let S_0 be a 2-sphere in the 3-cell C homeomorphically close to S such that $S_0 \cap S=X$, $S_0 \cap \bigcup A_i=S \cap \bigcup A_i$, and S_0 is locally tame except possibly at X (intuitively, to obtain S_0 we deform S into Int C along the components of $S-X$). By [6, Theorem 8.2], S_0 is tame, and hence has a cartesian product neighborhood N (i.e., one homeomorphic to $S_0 \times [-1, 1]$, with $S_0 \times \{0\}$ corresponding to S_0).

By [14, Theorem 1], there is a homeomorphism of N onto $S_0 \times [-1, 1]$ as above with some open arc of $\bigcup A_i$ about each p_j corresponding to the product of p_j with some open interval about 0. Thus there is a tame 2-sphere S_1 in N such that $S_1 \cap S_0=J$, the two components of S_1-J are in different components of $N-S_0$,

and S_1 contains the closure of an open arc of $\bigcup A_i$ about each p_j, and otherwise misses $\bigcup A_i$.

Note that $S_1 \cap X = J$ and that each point of J is an inaccessible point of X. Hence, if the components of $S - X$ are $K_1, K_2, \ldots$, then $S_1 \cap S$ consists of J plus the *null* sequence of *compact* sets: $S_1 \cap K_1$, $S_1 \cap K_2$, $S_1 \cap K_3, \ldots$. Since each $S_1 \cap K_i$ can be covered by the interiors of a finite number of disjoint 2-cells lying in S_1 and having boundaries close to $S_1 \cap K_i$, it follows that there is a disjoint, null sequence of 2-cells in $S_1 - J$ whose interiors cover $S_1 \cap (S - J)$. Thus, there is a (tame) 2-cell G in S_1 within $\varepsilon/2$ of J and missing the closure of $B - C$ such that $G \cap J$ is a spanning arc of G containing $p_1, \ldots, p_m$ in its interior, the two components of $(\text{Bd } G) - J$ are in different components of $E^3 - S$, and $G \cap \bigcup A_i$ is a collection of spanning arcs of G, each joining one component of $(\text{Bd } G) - J$ to the other.

Let $h_1 : E^3 \to E^3$ be a homeomorphism which is the identity outside a small neighborhood of the topological half plane $(\text{Int } G) - (\text{Int } C)$ and such that, for some neighborhood U of the open arc $\text{Int } (G \cap J)$, $h_1(U \cup [(\text{Int } G) - C]) \subset E^3 - C$. We may think of h_1 as squeezing the 2-cell $G - \text{Int } C$ very close to the arc $(\text{Bd } G) - \text{Int } C$, while keeping this arc fixed. Now, using the fact that C is a 3-cell, let $h_2 : E^3 \to E^3$ be a homeomorphism which is the identity on $E^3 - \text{Int } C$ and such that

$$h_2(C \cap \bigcup A_i) \subset U.$$

The required homeomorphism is then defined to be $h = h_1 h_2$.

COROLLARY 1.1. *Let K be a finite complex and L a subcomplex of K such that K contracts to L. Let M be a piecewise-linear 3-manifold, $g : K \to \text{Int } M$ a topological embedding, and $A_1, A_2, \ldots, A_k$, a finite disjoint collection of tame arcs in $M - g(L)$ with each $\text{Bd } A_j \subset M - g(K)$. Then, for given $\varepsilon > 0$, there is a homeomorphism $h : M \to M$ such that each $h(A_i) \subset M - g(K)$ and h is the identity on $g(L)$ and outside the ε-neighborhood of a compact set in $g(K) - g(L)$.*

Proof. We first take a barycentric subdivision of K so fine that the g-image of each simplex of this subdivision lies in an open 3-cell in M (this will not destroy the contraction hypothesis, by [15, Theorem 4]). We then use Theorem 1 and induction on the number of elementary expansions required (after the subdivision) to expand L into K. The details are routine, and will be omitted.

COROLLARY 1.2. *Theorem 1 is valid when, in its statement, E^3 is everywhere replaced by an arbitrary piecewise-linear 3-manifold M such that $B \subset \text{Int } M$.*

Proof. Let C and E have the same meaning as in the first paragraph of the proof of Theorem 1. Then there is a homeomorphism of $[0, 1]^q$ onto B under which $[0, 1]^{q-1} \times [0, 1/2]$ corresponds to the closure of $B - C$. The result then follows from Corollary 1.1 and the fact that some subdivision K of $[0, 1]^q$ contains some subdivision L of $[0, 1]^{q-1} \times [0, 1/2]$ as a subcomplex, with K contracting to L.

COROLLARY 1.3. *Let K be a finite complex, L a subcomplex of K such that K contracts to L, and g: K → Int M a homeomorphism, where M is a piecewise-linear 3-manifold. Then, if g(L) has the cube-with-handles property, so does g(K).*

Proof. We may assume that M is a compact 3-manifold-with-boundary, and hence an absolute neighborhood retract. There exists $\eta > 0$ such that any two mappings of S^1 into M which differ by less than η are homotopic in M. Let U be a neighborhood of $g(K)$ in M and r a retraction of U onto $g(K)$ such that r differs from the inclusion: $U → M$ by less than η.

We claim that U is an orientable neighborhood of $g(K)$. For, any loop $f: S^1 → U$ is homotopic in M to the loop $rf: S^1 → g(K)$, hence to a loop in $g(L)$, and hence is orientation-preserving.

We may now apply the criterion of Lemma 2 to $g(K)$. Let $\varepsilon > 0$ and a polygonal arc A be given such that $g(K) \cap \mathrm{Bd}\, A = \varnothing$. Since $g(L)$ has the cube-with-handles property, there is a homeomorphism $h_1: M → M$ which is the identity outside the ε-neighborhood of $g(L)$ and such that $g(L) \cap h_1(A) = \varnothing$. By Corollary 1.1, there is a homeomorphism $h_2: M → M$ which is the identity outside the ε-neighborhood of $g(K)$ and such that $g(K) \cap h_2 h_1(A) = \varnothing$. Then $h = h_2 h_1$ is the required homeomorphism.

REMARK. It follows from Corollary 1.3 and the corollary to Lemma 3, that collapsible complexes and cubes-with-handles have the cube-with-handles property under every embedding in the interior of a 3-manifold M. If M is orientable, then every homeomorphic image in Int M of a compact 2-manifold with nonempty boundary has the cube-with-handles property.

COROLLARY 1.4. *Let M be a piecewise-linear 3-manifold, and X a collapsible complex topologically embedded in Int M. Then X is cellular with respect to piecewise-linear cells in M if and only if: for each open set U of M containing X, there is an open set V of M such that $X \subset V \subset U$ and each loop in $V - X$ is contractible in $U - X$.*

Proof. This is immediate from the previous remark and [12, Theorem 1′].

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THE UNIVERSITY OF VIRGINIA,
       CHARLOTTESVILLE, VIRGINIA